Stochastic modeling of coupled electromechanical interaction for uncertainty quantification in electrostatically actuated MEMS

Nitin Agarwal, N.R. Aluru *

Department of Mechanical Science and Engineering, Beckman Institute for Advanced Science and Technology, University of Illinois at Urbana-Champaign, 405 N. Mathews Avenue, Urbana, IL 61801, United States

Received 20 August 2007; received in revised form 11 December 2007; accepted 16 January 2008
Available online 26 January 2008

Abstract

This work proposes a stochastic framework based on generalized polynomial chaos (GPC), to handle uncertain coupled electromechanical interaction, arising from variations in material properties and geometrical parameters such as gap between the microstructures, applicable to the static analysis of electrostatic MEMS. The proposed framework comprises of two components – a stochastic mechanical analysis, which quantifies the uncertainty associated with the deformation of MEM structures due to the variations in material properties and/or applied traction, and a stochastic electrostatic analysis to quantify the uncertainty in the electrostatic pressure due to variations in geometrical parameters or uncertain deformation of the conductors. The stochastic analysis is based on a stochastic Lagrangian approach, where, in addition to uncertain input parameters and unknown field variables, the random deformed configuration is expanded in terms of GPC basis functions. The spectral modes for the unknown field variables are finally obtained using Galerkin projection in the space spanned by GPC basis functions. The stochastic mechanical and electrostatic analyses are performed in a self-consistent manner to obtain the random deformation of the MEM structures. Various numerical examples are presented to study the effect of uncertain parameters on performance of various MEMS devices. The results obtained using the proposed method are verified using rigorous Monte Carlo simulations. It has been shown that the proposed method accurately predicts the statistics and probability density functions of various relevant parameters.

© 2008 Elsevier B.V. All rights reserved.

Keywords: Multiphysics; Spectral stochastic boundary element method (SSBEM); Spectral stochastic finite element method (SSFEM); Polynomial chaos; Lagrangian electrostatic analysis; Large deformation; Geometrical uncertainty; Uncertainty propagation

1. Introduction

The design and analysis of micro-electromechanical systems (MEMS) requires considering the interaction of various physical fields such as mechanical, electrical and possibly fluidic. In recent years, the advances in numerical simulation methods have increased the ability to accurately model these devices [1–4]. These simulation methods assume that the material properties and various geometrical parameters of the device are known in a deterministic sense. However, low cost manufacturing processes used for MEMS often result in significant uncertainties in these parameters which may lead to large variation in the device performance. Thus, in order to design reliable and efficient electrostatic MEMS devices, it is required to consider a stochastic analysis of the coupled electromechanical interaction.

The stochastic analysis of coupled electromechanical problem is complicated by two facts – firstly, due to the nature of the coupling between the mechanical and electrostatic energy domains, we need to deal with geometrical variations during the analysis, which is not straightforward, as it leads to random computational domains. Secondly, because of the multiphysics nature of the problem, uncertainties propagate from one energy domain to
another, and need to be considered carefully. A framework for quantifying uncertainty for the electroosmotic and pressure-driven flow in a microchannel, involving coupled transport and electrostatic field equations, has been presented in [5]. In the past, the stochastic variations in various parameters during the design of electrostatic MEMS has been addressed using Monte Carlo (MC) simulations. The effect of various geometrical features on the design of a comb drive has been studied in [6] using Monte Carlo method incorporated in the ANSYS probabilistic design system (ANSYS/PDS). The variability in the performance of a ceramic MEMS actuator resulting from variations in the shape of the actuator and the air gap in the condenser, has been studied using MC simulations in [7]. A reliability-based analysis using the first-order reliability method (FORM) and design optimization framework for electrostatic MEMS is presented in [8]. Since the MC based methods are statistical in nature, their accuracy depends on the sample size. The simulations may become prohibitively expensive, especially for the analysis of MEMS devices, as it is expensive to solve the coupled electromechanical problem even in the deterministic case.

The most widely used non-statistical stochastic analysis technique is based on the approach pioneered by Ghanem and Spanos [9], known as polynomial chaos. The basic idea is to treat the uncertainty as a separate dimension (in addition to space and time) and to expand the field variables along the random dimension using polynomial chaos basis functions. Polynomial chaos is essentially a spectral expansion of the stochastic processes in terms of the orthogonal polynomials as given by Wiener’s homogeneous chaos theory [10]. The homogeneous chaos expansion is based on Hermite polynomials and converges exponentially to any second-order random process, when the underlying random variables are Gaussian. This idea was further generalized by Xiu and Karniadakis [11], to obtain exponentially converging algorithms even for non-Gaussian random variables. This stochastic analysis technique based on polynomial chaos has been applied to model uncertainty in various problems such as computational mechanics [12,13], diffusion [14], fluid flow [15–17] and heat conduction [18,19].

This work presents a stochastic framework based on generalized polynomial chaos, to handle uncertain coupled electromechanical interaction, arising from variations in material properties and geometrical parameters such as gap between the microstructures, applicable to the static analysis of electrostatic MEMS. The proposed framework comprises of two components – a stochastic mechanical analysis, which quantifies the uncertainty associated with the deformation of MEM structures due to the variations in material properties and/or applied traction, and a stochastic electrostatic analysis to quantify the uncertainty in the electrostatic pressure due to variations in geometrical parameters or uncertain deformation of the conductors.

The stochastic analysis is based on a stochastic Lagrangian approach, where the random deformed configuration is expanded as a sum of its mean, known as the mean or initial undeformed configuration, and a random deformation field applied to the conductors defined by the mean (or undeformed) configuration. For the stochastic mechanical analysis, in addition to the random deformation field, the uncertain input parameters such as Young’s modulus, applied traction, etc. are expanded in terms of the generalized polynomial chaos (GPC) basis functions. These expansions are then used to compute stochastic analogues of various quantities that appear in the large deformation analysis. The uncertain deformation is finally obtained using Galerkin projection in the space spanned by GPC basis functions. The spatial discretization is done using standard finite element method (FEM). This methodology for quantifying uncertainty in large deformation problems is presented in [20], and has been applied to elastoplasticity problems. For the stochastic electrostatic analysis, the random deformation field is used to obtain a stochastic Lagrangian boundary integral equation [21], which can be solved for uncertain surface charge density. The random surface charge density is discretized both in the random dimension and space using GPC and classical boundary element method (BEM), respectively. The computed uncertain surface charge density is then used to obtain the random electrostatic pressure. These stochastic mechanical and electrostatic analyses are performed in a self-consistent manner to obtain the random deformation of the MEM structures. We consider several numerical examples to study the effect of uncertain parameters on performance of various MEMS devices, highlighting the features of the proposed methodology. The results obtained using the proposed method are also verified using rigorous Monte Carlo simulations. It has been shown that the proposed method accurately predicts the statistics and probability density functions of various relevant parameters.

The paper is organized as follows: in Section 2 we present the most widely used spectral expansion method for random fields – the generalized polynomial chaos (GPC) expansion and various techniques required to employ GPC expansion as a stochastic discretization technique. In Section 3 we present the deterministic coupled electromechanical problem. In Section 4 we then present the stochastic electrostatic and mechanical analysis based on the stochastic Lagrangian approach and finally describe the formulation for the stochastic coupled electromechanical problem. In Section 5 we present some numerical examples to demonstrate the proposed methodology to quantify the effect of uncertain material properties and geometrical parameters on the performance of electrostatic MEMS devices. We finally conclude the discussion in Section 6.

2. Spectral stochastic representation

Let $D$ be a domain in $\mathbb{R}^d, d = 1, 2$ and $x \in D$. Let $(\Omega, B, \mathcal{P})$ denote a probability space, where $\Omega$ is the set of elementary events, $B$ is the $\sigma$-algebra of events and $\mathcal{P}$ is the probability measure. The symbol $\theta$ specifies an
elementary event in $\Theta$ and in the following presentation any quantity with $\theta$-dependence denotes a random quantity. Then, all real valued functions $\xi(\theta)$ defined on $\Theta$ are known as random variables and functions $w(x, \theta)$ defined on $D \times \Theta$ are known as random fields or processes. Uncertainties can be described using these stochastic quantities – uncertain parameters are modeled as random variables and uncertain spatial functions are represented as random fields.

From a numerical viewpoint, the random fields need to be discretized both in the random dimension and the spatial dimension $x$. Thus, we seek a stochastic discretization that represents a random field in terms of finite number of independent random variables. The most widely used method for the discretization of random fields is generalized polynomial chaos expansion (GPCE) [9,11], which is described next.

### 2.1. Generalized polynomial chaos expansion (GPCE)

The polynomial chaos expansion is essentially a spectral expansion of the random field in terms of the orthogonal polynomials in multi-dimensional random variables. The original polynomial chaos, as first proposed by Wiener [10], employs Hermite polynomials in terms of Gaussian random variables, and such an expansion converges to any $L_2$ function in the probability space in accordance with the Cameron–Martin theorem [22]. Although in theory, the Hermite chaos can be used to represent arbitrary random process, it achieves optimal exponential convergence rates only for Gaussian or near Gaussian random fields. A more general framework, known as the generalized polynomial chaos was developed in [11]. The GPCE for a second-order random process or field $w(x, \theta)$ is given as:

$$w(x, \theta) = \sum_{i=0}^{\infty} w_i(x) \Psi_i(\xi(\theta)), \quad (1)$$

where the coefficients $\{w_i\}$ are deterministic and the polynomials $\{\Psi_i\}$ are chosen from the hypergeometric polynomials of the Askey scheme. The type of the random variables is selected based on the random inputs, and their weighting function in turn decides the type of orthogonal polynomials used as basis in the chaos expansion.

The functions $\{\Psi_i\}$ form an orthogonal basis in the probability space, with the orthogonality relation

$$\langle \Psi_i, \Psi_j \rangle = \delta_{ij} \langle \Psi_i^2 \rangle, \quad (2)$$

where $\delta_{ij}$ is the Kronecker delta and $\langle \cdot, \cdot \rangle$ denotes the ensemble average which is the inner product given as

$$\langle \Psi_i, \Psi_j \rangle = \int_{\Omega} \Psi_i(\xi) \Psi_j(\xi) d\mathcal{P} \quad (3)$$

We note that the summation in Eq. (1) is infinite and that the PC basis functions $\{\Psi_i\}$ are functions of the elements of the infinite set $\{\xi_i(\theta)\}_{i=1}^{\infty}$. However, in practice it is logical to use a finite-dimensional set $\{\xi_i(\theta)\}_{i=1}^{n}$, which yields an $n$-dimensional polynomial chaos expansion. Also, we only consider polynomials up to some finite order $p$. Thus, the expansion in Eq. (1) can now be written as,

$$w(x, \theta) = \sum_{i=0}^{N} w_i(x) \Psi_i(\xi(\theta)). \quad (4)$$

The total number of terms included in the polynomial chaos expansion is given as $(N+1) = \binom{n+p}{n}$, which depends both on the dimensionality $n$ and the highest order $p$ of the multi-dimensional polynomials used.

For a random field $w(x, \theta)$ with PC expansion given by Eq. (4), the mean $\bar{w}(x)$ and standard deviation $\sigma(x)$ can be computed as,

$$\bar{w}(x) = w_0(x), \quad \sigma^2(x) = \sum_{i=1}^{N} w_i^2(x) \langle \Psi_i^2 \rangle, \quad (5)$$

using the orthogonality relation as given by Eq. (2).

### 2.2. Operations on PC variables

In the spectral stochastic uncertainty quantification approach, the model parameters and unknown variables are first expanded in terms of PC basis functions and are substituted into the governing equations. A Galerkin projection is then used to solve for the coefficients in the PC expansion of the unknown variables. Thus, all the mathematical operations that were previously performed on the variables in governing equations, now need to be performed on corresponding random quantities, given in the form of PC expansion. The elementary arithmetic operations such as addition, subtraction, multiplication and division for random scalars are presented in [23], and the corresponding matrix operations are given in [20]. Some of these operations which are used in later sections are presented here.

Given two matrices $A = A_i \Psi_i$ and $B = B_j \Psi_j$ (where summation is implied),

- Matrix product can be given as $C = C_k \Psi_k = (A_i \Psi_i) \cdot (B_j \Psi_j)$, where $C_k$ can be evaluated using Galerkin projection as,

$$C_k = \frac{A_i B_j \epsilon_{ijk}}{d_{ik}}, \quad (6)$$

where, the quantities $\epsilon_{ijk} = \langle \Psi_i \Psi_j \Psi_k \rangle$ and $d_{ik} = \langle \Psi_i^2 \rangle$ can be pre-computed and stored for later use. The product of more than two matrices can be computed using a pseudo-spectral approach, where we repeatedly use the product of two matrices.
- Matrix transpose can simply be computed by transposing the corresponding modes,

$$A^T = A_i^T \Psi_i. \quad (7)$$
- Determinant of a matrix can be computed using the multiplication of random scalars.
• Other complicated functions can be evaluated using a direct integration procedure. Suppose, we need to express $\mathcal{F}(x)$ in terms of PC bases, where $\mathcal{F}$ is a nonlinear function of a given random parameter $x = x_i \psi_i$. We can write

$$\mathcal{F}(x) = \sum_{k=0}^{N} \mathcal{F}_k \psi_k^2, \quad \mathcal{F}_k = \langle \mathcal{F}(x), \psi_k^2 \rangle,$$

where the integral $\langle \mathcal{F}(x), \psi_k^2 \rangle$ can be computed using an appropriate numerical quadrature rule.

We must note that, although the direct integration technique leads to accurate results, it may be expensive as the number of random dimensions increases or when such projections need to be computed repeatedly. For certain nonlinear functions, such as square roots, logarithm, etc., it may be possible to still perform these projections in a reasonably efficient manner, by using some of the techniques for non-polynomial function evaluation given in [23].

3. Deterministic coupled electromechanical problem

Physical level analysis of electrostatic MEMS requires a self-consistent solution of the coupled mechanical and electrostatic equations. Fig. 1 shows a typical MEM device – a deformable cantilever beam over a fixed ground plane. When a potential difference is applied between the beam and the ground plane, it induces electrostatic charge on the surface of the conductors. This charge gives rise to an electrostatic pressure acting normal to the surface of the beam, which causes the beam to deflect. When the beam deflects, the charge redistributes on the surface of the conductors, and consequently the electrostatic force and the deflection of the beam also change. This explains the coupling between mechanical and electrostatic energy domains, and a self-consistent solution is required to obtain the final deflection of the beam.

A framework for the deterministic analysis of electrostatic MEMS is presented in [3], which uses a Lagrangian description both for the mechanical and the electrostatic domains. The mechanical deformation of the beam is obtained by performing a 2-D geometrically nonlinear elasticity analysis [24]. The governing equations for the deformation of an elastic body in the absence of body force are given as

$$\nabla \cdot (FS) = 0 \quad \text{in } \Omega,$$

$$u = G \quad \text{on } \Gamma_g,$$

$$P \cdot N = H \quad \text{on } \Gamma_h,$$

where $\Omega$ represents the undeformed configuration with boundary $d\Omega = \Gamma_g \cup \Gamma_h$. $u$ is the displacement vector, $F$ is the deformation gradient, $P$ and $S$ are the first and second Piola–Kirchhoff stress tensors, respectively. $H$ is the electrostatic pressure acting on the surface of the structures and $N$ is the unit outward normal vector in the undeformed configuration. The prescribed displacement is given by $G$. The constitutive law can be written as

$$S = \mathcal{C} E,$$

$$E = \frac{1}{2} (F^T F - I),$$

where $\mathcal{C}$ is the material tensor and $E$ is the Green–Lagrangian strain.

The electrostatic analysis is done using a Lagrangian boundary integral form as described in [25]. The Lagrangian boundary integral equations are given by:

$$\phi(P) = \int_{d\Omega} G(p(P),q(Q)) \sigma(q(Q)) [T(Q) \cdot C(Q) T(Q)]^{1/2} d\Gamma_Q + C,$$

$$C_T = \int_{d\Omega} \sigma(q(Q)) [T(Q) \cdot C(Q) T(Q)]^{1/2} d\Gamma_Q,$$

where $\sigma$ is the unknown surface charge density and $C$ is an unknown constant that needs to be computed. $P$ and $Q$ refer to the positions of source and field points, respectively, in the undeformed configuration. $G$ is the Green’s function, which in 2-D is given as $G(P,Q) = -\frac{1}{2\pi} \ln |P - Q + u_0|$, where $\epsilon$ is the dielectric constant of the medium and $|P - Q + u_0|$ represents the distance between the source and field points in the deformed configuration. $C_T$ represents the total charge of the system, which is set to be zero. $C(Q) = \epsilon T(Q) F(Q)$ and $T(Q)$ is the unit tangential vector at the field point $Q$ in the undeformed configuration. The electrostatic pressure acting on the conductors in the undeformed configuration can be computed from the surface charge density as,

$$H = P_e J F^T N,$$

where $P_e = \frac{\epsilon}{2}$ is the electrostatic pressure acting normal to the surface of the conductors and $J = \det(F)$. Eq. (16) represents the nonlinear coupling between the mechanical and electrostatic energy domains, and a relaxation or Newton scheme can be used to obtain self-consistent solutions, as described in [3].

4. Stochastic coupled electromechanical problem

The simulation methods developed for MEMS assume that the geometrical and material properties of the device are known in a deterministic sense. However, in reality, these devices are subjected to severe stochastic variations

![Fig. 1. MEM beam under deformation.](image-url)
in various parameters, which may affect their performance. In order to quantify the effect of these stochastic variations in material properties and geometrical parameters on the behavior of MEMS devices, a stochastic analysis of the coupled electromechanical interaction is required. The nature of the electromechanical coupling suggests that the uncertainties from the mechanical energy domain are propagated into electrostatics, only via an uncertain deformation field. Similarly, the variations in the electrostatic energy domain results in a stochastic electrostatic pressure, which needs to be considered during the mechanical analysis. Thus, the stochastic modeling of coupled electromechanical interaction, aimed at capturing the variations in material properties and geometrical parameters, requires two components – a stochastic electrostatic analysis, which quantifies the effect of uncertain material properties and/or loading, on the deformation field, and a stochastic mechanical analysis, which quantifies the effect of uncertain material properties and/or loading, on the deformation field.

The methodology for quantifying uncertainty propagation in large deformation problems is presented in [20], and has been applied to elasto-plasticity problems. A framework to handle geometrical variations in electrostatic analysis is developed in [21]. Both of these frameworks employ a stochastic Lagrangian concept, where the deformation of the body is modeled as a random field, which is described next.

4.1. Stochastic Lagrangian formulation

In this section we consider the stochastic deformation of an elastic body. As shown in Fig. 2, we represent the initial undeformed configuration of an elastic body as \( \Omega \), and let \( \mathbf{X} \) denote the co-ordinates of a material point in \( \Omega \). The deformation field \( \mathbf{u}(\mathbf{X}, \theta) \) is modeled as a random field which depends both on space and the random dimension. As described in Section 2, we expand the displacement field, in terms of the GPC basis functions as

\[
\mathbf{u}(\mathbf{X}, \theta) = \sum_{m=0}^{M} \mathbf{u}_m(\mathbf{X}) \Psi_m(\theta),
\]

where \( M + 1 \) is the total number of terms considered in the GPC expansion. The elastic body undergoes a stochastic deformation, such that the position of the material point, in the random deformed configuration \( \omega(\theta) \) is given as

\[
\mathbf{x}(\theta) = \mathbf{X} + \mathbf{u}(\mathbf{X}, \theta).
\]

The stochastic deformation gradient \( \mathbf{F}(\mathbf{X}, \theta) \) can be given as

\[
F_{ij} = \frac{\partial X_i}{\partial X_j} = \delta_{ij} + \sum_{m=0}^{M} \frac{\partial u_{m,i}}{\partial X_j} \Psi_m(\theta), \quad i, j = 1, 2 \quad \text{for 2D},
\]

where \( u_{m,i} \) denotes the \( i \)th component of the \( m \)th displacement mode vector \( \mathbf{u}_m \). We must note that, similar to the deterministic case, the constraint \( \det \mathbf{F}(\theta) > 0 \) must be true, in order to have meaningful deformation. From Eq. (19) one can easily see that \( \mathbf{F}(\theta) \) can be written as

\[
\mathbf{F}(\theta) = \sum_{i=0}^{M} \mathbf{F}_i(\mathbf{X}) \Psi_i(\theta).
\]

We now use the stochastic Lagrangian concept to present the stochastic framework for electrostatics and large deformation elasticity analysis.

4.2. Stochastic electrostatic analysis

The Lagrangian boundary integral equations given by Eqs. (14) and (15) allow to solve for the surface charge density in the deformed configuration by solving the electrostatic equations on the undeformed configuration. In this section, we present the analogous stochastic formulation to compute the uncertain surface charge density, by solving the electrostatic equations on the deterministic undeformed geometry.

4.2.1. Stochastic Lagrangian boundary integral equations

We employ the stochastic Lagrangian formulation, where the displacement is modeled as a random field, and is expressed in terms of GPC basis functions, as given by Eq. (17). The position of a point on the uncertain boundary \( \mathbf{x}(\theta) \) is given by Eq. (18), using which the stochastic Green’s function in two dimensions \( G(p, q, \theta) \) can be written as

\[
G(p, q, \theta) = G(p(P), q(Q), \theta)
\]

\[
= -\frac{1}{2\pi\varepsilon} \ln |P - Q + \mathbf{u}_P(\theta) - \mathbf{u}_Q(\theta)|,
\]

where \( P \) and \( Q \) are the source and field points in the mean configuration, \( p \) and \( q \) are the source and field points in the uncertain configuration, and \( \mathbf{u}_P(\theta) \) and \( \mathbf{u}_Q(\theta) \) are the random displacements associated with \( P \) and \( Q \), respectively.

Fig. 2. Stochastic deformation.
Using Eq. (21) we can write the Stochastic Lagrangian boundary integral equations for electrostatics, analogous to Eqs. (14) and (15) for the deterministic case as follows:

\[
\phi(P) = \int_{d\Omega} G(p(P), q(Q), 0) \sigma(q(Q), 0) \gamma(Q, 0) d\Gamma_Q + C(0),
\]

\[
C_T = \int_{d\Omega} \sigma(q(Q), 0) \gamma(Q, 0) d\Gamma_Q,
\]

where \(d\Omega = \bigcup_{i=1}^{N_s} d\Omega_i\) such that \(d\Omega_i\) represents the deterministic undeformed configuration for \(i\)th conductor and \(N_c\) is the total number of conductors. \(C(0)\) is an unknown random variable and \(C_T\) is the total charge, which is set to be zero. \(\gamma(Q, 0) = \left| \mathbf{T}(Q) \cdot \mathbf{C}(Q, 0) \mathbf{T}(Q) \right|^2\), where \(\mathbf{C}(Q, 0) = \mathbf{F}^T(Q) \mathbf{F}(Q)\) is the stochastic Green deformation tensor and \(\mathbf{T}(Q)\) is the tangent at the field point \(Q\). We note that all the integrals in Eqs. (22) and (23) are defined over the undeformed configuration \(d\Omega\) and all the quantities inside the integrals are appropriately defined in terms of the quantities in the undeformed configuration.

4.2.2. Stochastic discretization – generalized polynomial chaos

To solve the stochastic equations, in addition to the spatial discretization we also need to consider the discretization with respect to the random dimension \(\theta\). We write the polynomial chaos expansion for the unknown surface charge density \(\sigma(x, \theta)\), and the random variable \(C(0)\) as follows:

\[
\sigma(x, \theta) = \sum_{n=0}^{N} \sigma_n(x) \Psi_n(\xi(\theta)), \quad C(0) = \sum_{n=0}^{N} C_n \Psi_n(\xi(\theta)),
\]

where \((N + 1)\) is the total number of terms considered in the truncated polynomial chaos expansion. We note that the uncertainty associated with the surface charge density is included in the polynomial basis \(\Psi(\xi(\theta))\) and hence the spectral modes \(\sigma_n(x)\) and \(C_n, n = 0, \ldots, N\) are deterministic. We also express \(\phi(P, Q, \theta) = G(p(P), q(Q), 0) \gamma(Q, 0)\) and \(\gamma(Q, 0)\) in terms of the GPC basis functions \(\{\Psi_m\}\) as explained in [21],

\[
\phi(P, Q, \theta) = \sum_{l=0}^{N} \phi_l(P, Q) \Psi_l(\xi(\theta)),
\]

\[
\gamma(Q, 0) = \sum_{l=0}^{N} \gamma_l(Q) \Psi_l(\xi(\theta)).
\]

Using expansions given by Eqs. (24) and (25), in Eqs. (22) and (23) and replacing \(\Psi_n(\xi(\theta))\) by \(\Psi_n\) for clarity, and performing Galerkin projection onto GPC basis functions \(\{\Psi_m, m = 0, \ldots, N\}\), we get,

\[
\phi(P) \langle \Psi_m \rangle = \int_{d\Omega} \sum_{n=0}^{N} \sum_{l=0}^{N} \phi_l(P, Q) \sigma_n(q(Q)) \langle \Psi_m \rangle,
\]

\[
\langle \Psi_m \Psi_n \rangle d\Gamma_Q + C_m \langle \Psi_m^2 \rangle \quad m = 0, \ldots, N,
\]

\[
C_T \langle \Psi_m \rangle = \int_{d\Omega} \sum_{n=0}^{N} \sum_{l=0}^{N} \gamma_l(Q) \sigma_n(q(Q)) \langle \Psi_m \rangle,
\]

\[
\langle \Psi_m \Psi_n \rangle d\Gamma_Q \quad m = 0, \ldots, N,
\]

\[
\text{using the orthogonality relation } \langle \Psi_m, \Psi_n \rangle = \delta_{mn} \langle \Psi_m^2 \rangle,
\]

where \(\langle \cdot \rangle = \langle \cdot, 1 \rangle\) represents the ensemble average. The above set of equations represent \(2(N + 1)\) coupled integral equations that need to be solved for \(2(N + 1)\) unknowns, \(\sigma_n(x), C_n, n = 0, \ldots, N\). We note that these equations are deterministic as the unknowns \(\{\sigma_n(x)\}\) depend only on \(x\) and \(\{C_n\}\) are constants.

The set of equations given by Eqs. (26) and (27), can be discretized in space using classical Boundary Element Method (BEM) [26], where the surface of the conductors is discretized into segments or panels. Further, we assume that each of \(\sigma_n(x)\) is constant over each panel, such that

\[
\sigma_n(x(X)) = \sum_{k=1}^{K} \sigma_n N_k(X) \quad n = 0, \ldots, N,
\]

where \(N_k(X), k = 1, \ldots, K\) are the piecewise constant shape functions used for the collocation method and \(K\) is the total number of panels. This leads to \(N_s \times N_l\) linear system as detailed in [21], where \(N_l = (K + 1)(N + 1)\), which can be solved to obtain the spectral modes \(\sigma_n(x), n \in [0, N]\). Once we obtain the spectral modes \(\{\sigma_n\}\), the surface charge density is obtained using Eq. (24):

\[
\sigma(x, \theta) = \sum_{n=0}^{N} \sigma_n(x) \Psi_n(\xi(\theta)).
\]

Also, using Eq. (16) the electrostatic pressure in the undeformed configuration \(\mathbf{H}(X, \theta)\), is given as:

\[
\mathbf{H}(X, \theta) = \frac{\sigma^2(x(X), \theta)}{2\varepsilon} J^T \mathbf{F}(X, \theta)^T \mathbf{N}(X),
\]

where \(J\) is the determinant of \(\mathbf{F}\) and \(\mathbf{N}(X)\) is the unit outward normal at \(X\) in the undeformed configuration.

4.2.3. Illustrative example

In this section, we present a numerical example to demonstrate the stochastic Lagrangian formulation to model geometrical variations in electrostatic analysis. We consider a cantilever beam of dimensions \(80 \times 1 \mu\text{m}^2\) located at a gap \(g\) over a grounded plane with an applied potential \(V = 1\, \text{V}\), as shown in Fig. 1. We wish to quantify the effect of variation in gap \(g\), on the surface charge density and the capacitance between the beam and the ground plane. The gap \(g\) is assumed to be a uniformly distributed random variable given as,

\[
g = g_0(1 + v_g \xi),
\]

where \(g_0 = 1\, \mu\text{m}\) is the mean gap, \(\xi\) is a uniformly distributed random variable on \([-1, 1]\) and \(v_g = 0.2\) which signi-
fies a 20% variation in the value of the gap. As can be seen easily, the deterministic undeformed configuration is given by the beam located at a gap \( g = g_0 \) above the ground plane. We must note that although the beam does not undergo any physical deformation in this case, the variation in the gap can still be modeled by applying a zero mean random displacement field \( u = [0, g_0, \xi]^T \) to the ground plane or the beam. We choose to apply this random displacement to the beam located at a gap \( g_0 \) from the ground plane. This displacement can also be identified as the spectral expansion given in Eq. (17) with only the second term being non-zero. We employ the generalized polynomial chaos expansion of order \( p \), based on one-dimensional Legendre polynomials and compute the spectral modes \( \{ \sigma_i(x) \} \). The mean surface charge density profile for the beam is plotted in Fig. 3a.

The capacitance between the beam and the ground plane can be computed as

\[
C_{bg}(\theta) = \frac{\partial(\theta)}{V},
\]

where \( \partial(\theta) \) is the total charge on the beam, which is given as

\[
\partial(\theta) = \int_{\Omega_{beam}} \sigma(x, \theta) [\mathbf{T} \cdot \mathbf{C} \mathbf{T}]^T \text{d}F.
\]

We plot the probability density function (PDF) of the capacitance using Monte Carlo (MC) simulations and generalized polynomial chaos (GPC) of several orders in Fig. 3b. For MC simulations we use 10,000 realizations of \( g \) (generated according to the uniform distribution), and corresponding to each realization the deterministic problem is solved and we obtain the capacitance values \( \{C_{bg}\} \). In order to get the PDFs using GPC, we generate realizations of \( \xi \) in accordance with the uniform distribution and obtain the values \( \{C_{bg}(\xi)\} \). A histogram of these \( \{C_{bg}\} \) values is then generated based on 10 equally spaced bins and the probability density is obtained by normalizing the frequency corresponding to each bin with the bin size and the total number of realizations. The PDFs are finally generated by plotting these probability density values along the centers of the bins.

As can be seen from Fig. 3b, there is a good agreement between the PDFs obtained using MC simulations and GPC. Also, we notice that the PDF obtained using GPC converges to the MC result, as the order of expansion is increased from \( p = 2 \) to \( p = 3 \). The mean and standard deviation values for capacitance between the beam and the ground plane, as obtained using MC and GPC are given in Table 1.

### Table 1

<table>
<thead>
<tr>
<th></th>
<th>MC</th>
<th>GPC2</th>
<th>GPC3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean ( \times 10^{-2} ) (pF/m)</td>
<td>7.9332</td>
<td>7.9332</td>
<td>7.9331</td>
</tr>
<tr>
<td>Std ( \times 10^{-2} ) (pF/m)</td>
<td>0.9372</td>
<td>0.9360</td>
<td>0.9366</td>
</tr>
</tbody>
</table>

4.3. Stochastic mechanical analysis

The governing equation, boundary conditions and constitutive law for the deterministic deformation of an elastic body in the absence of body force are given by Eqs. (9)–(12), respectively. Following the strong form as given by Eq. (9), the weak form can be written as follows:

Find \( u \in \mathcal{V} \) such that \( G(\mathbf{u}, \eta) = 0 \ \forall \eta \in \mathcal{W} \), where \( \mathcal{V} \) and \( \mathcal{W} \) are appropriately defined spaces of admissible displacements and test functions respectively, and

\[
G(\mathbf{u}, \eta) = \int_{\Omega} \nabla \eta : [(D \mathbf{F}) S] \text{d}\Omega - \int_{\Gamma_h} \eta H, \text{d}F, \quad (33)
\]

where \( x = \mathbf{X} = \mathbf{X} + \mathbf{u}(\mathbf{X}) \) and \( \mathbf{F} = D \mathbf{\Phi} \), such that \( x \) and \( \mathbf{X} \) represent the coordinates of a point in the deformed and the undeformed configuration, respectively. The Galerkin form can simply be obtained by replacing \( \mathbf{\Phi}, \eta \) and \( \mathbf{u} \) by

![Fig. 3. Effect of uncertain gap on surface charge density and capacitance of a beam over a ground plane, \( [v_g = 0.2, V = 1 \text{V}] \).](image-url)
\( \Phi^h, \eta^h \) and \( u^h \), respectively. We use the following approximations:

\[
\begin{align*}
\Phi^h &= \sum_{a=1}^{N_d} N_a d_a, \\
\eta^h &= \sum_{a=1}^{N_d} N_a c_a, \\
\Phi^h &= \sum_{a=1}^{N_d} N_a x_a, \\
\end{align*}
\]  
(34)

where \( \{N_a\} \) are the FEM basis functions used and \( N_d \) is the number of nodes in the finite element mesh. Using the linearization we eventually need to solve the following system:

\[
K \Delta d = -R(d),
\]
(35)

where \( K \) is the \( 2N_d \times 2N_d \) stiffness matrix, \( \Delta d \) and \( R(d) \) are the \( 2N_d \times 1 \) incremental nodal displacement and residual vectors, respectively. The details about computing these quantities are included in Appendix A, as it is also required for the stochastic analysis. The final displacement is computed by starting with some initial guess \( d = d^0 \), and solving the linear system given by Eq. (35) until convergence, such that

\[
d^{n+1} = d^n + \Delta d^n,
\]
(36)

where \( n \) refers to the \( n \)th Newton iteration.

4.3.1. Weak formulation for stochastic large deformation

For the stochastic case, in the strong form given by Eq. (9), all the quantities are not only functions of space \( X \) but also of the random dimension. We define \( \mathcal{F} = \text{span}\{\Psi_i, i = 0, \ldots, N\} \), where \( \Psi_i \) are the GPC basis functions, using which the weak form can be written in a similar way as for the deterministic case.

Find \( u \in \mathcal{W} \times \mathcal{F} \) such that,

\[
\tilde{G}(\Phi, \eta) = 0, \quad \forall \eta \in \mathcal{W} \times \mathcal{F},
\]
(37)

where,

\[
\tilde{G}(\Phi, \eta) = \int_{\Omega} \int_{\Omega} \nabla \eta : [(D\Phi(X, \theta))S(X, \theta)]d\Omega d\Omega \eta
\]

\[
-\int_{\Gamma_h} \eta H_i(X, \theta) d\Gamma d\Omega \eta.
\]
(38)

4.3.2. Spatial discretization – FEM

Using the finite element basis functions \( \{N_a, a = 1, \ldots, N_d\} \), we approximate the displacement \( u(X, \theta) \) as follows:

\[
u(X, \theta) = \sum_{a=1}^{N_d} N_a(X) d^r(\theta),
\]
(39)

where \( d^r(\theta) = [u^r(\theta), v^r(\theta)]^T \) are random nodal displacement vectors. Using this approximation in (37) and using Galerkin projection, where we take the test functions as \( \varphi N_\beta, \beta = 1, \ldots, N_d \), such that \( \varphi(\theta) \in \mathcal{F} \), and subsequent linearization, we require that,

\[
\int_{\Omega} \varphi K \Delta d d\Omega = -\int_{\Omega} \varphi R(d) d\Omega, \quad \forall \varphi \in \mathcal{F}.
\]
(40)

where \( K \) is the random stiffness matrix, \( \Delta d \) is the random incremental nodal displacement vector and \( R(d) \) is the random residual vector. We note that all these quantities are functions of random dimension \( \theta \), and need to be discretized in the stochastic domain.

4.3.3. Stochastic discretization – generalized polynomial chaos

We now expand the random displacement vector \( d(\theta) \) and random incremental displacement vector \( \Delta d(\theta) \) using polynomial chaos, as follows:

\[
d(\theta) = \sum_{j=0}^{N} d_j \psi_j(\theta), \quad \Delta d(\theta) = \sum_{j=0}^{N} \Delta d_j \psi_j(\theta),
\]
(41)

where we note the vectors \( \{d_j\} \) and \( \{\Delta d_j\} \) are deterministic and need to be computed. Using the expansion for incremental displacement vector in Eq. (40) we get,

\[
\int_{\Omega} \varphi K \sum_{j=0}^{N} \Delta d_j \psi_j(\theta) d\Omega = -\int_{\Omega} \varphi R(d) d\Omega, \quad \forall \varphi \in \mathcal{F}.
\]
(42)

Using the techniques described in Section 2.2, we can expand \( K \) and \( R(d) \) in terms of the GPC basis functions, as detailed in Appendix A,

\[
K(\theta) = \sum_{i=0}^{N} K_i \psi_i(\theta), \quad R(d(\theta)) = \sum_{i=0}^{N} R_i \psi_i(\theta).
\]
(43)

Using Eq. (43) in Eq. (42), we get

\[
\int_{\Omega} \varphi K_i \psi_i \Delta d \psi_j d\Omega = -\int_{\Omega} \varphi R_i \psi_i d\Omega, \quad \forall \varphi \in \mathcal{F},
\]
(44)

where summation is implied. Further, setting \( \varphi = \{\psi_k, k = 0, \ldots, N\} \), we get

\[
K_i \Delta d_j \langle \psi_i, \psi_j \rangle = -R_i \langle \psi_k^2 \rangle, \quad \forall k \in [0, N].
\]
(45)

Finally, we solve the linear system for vectors \( \Delta d_j \). We define the augmented vectors \( d = [d_0^T, \ldots, d_N^T]^T \) and \( \Delta d = [\Delta d_0^T, \ldots, \Delta d_N^T]^T \). The final displacement is computed by starting with some initial guess \( d = d^0 \), and solving the linear system given by Eq. (45) until convergence, such that

\[
\hat{d}^{n+1} = \hat{d}^n + \Delta \hat{d}^n,
\]
(46)

where \( n \) refers to the \( n \)th Newton iteration.

![Fig. 4. Doubly clamped beam under deformation.](image-url)
4.3.4. Illustrative example

We now present a numerical example to demonstrate the methodology presented above. We consider a beam with length 80 µm, thickness 1 µm and width 10 µm, which is clamped at both ends as shown in Fig. 4. A uniform load \( P_c = 5 \times 10^3 \) N/m is applied onto the top surface, such that it always acts normal to the surface, similar to the electrostatic pressure. The Young’s modulus is assumed to be a uniformly distributed random variable given as,

\[
E = E_0(1 + v_E \xi),
\]

where \( E_0 = 169 \) GPa is the mean value, \( v_E = 0.2 \) which represents a 20% variation and \( \xi \) is a uniformly distributed random variable in \([-1,1]\). The Poisson’s ratio for this problem is set to be 0.3. For this problem, the displacement is expanded using one-dimensional PC expansion based on Legendre polynomials, i.e.,

\[
u = u_0 + u_1 \xi + u_2 \frac{1}{2}(3\xi^2 - 1) + u_3 \frac{1}{2}(5\xi^3 - 3\xi) + \ldots,
\]

where the modes \( u_i \) are obtained using Eqs. (45) and (46).

In Fig. 5a we plot the probability density function (PDF) of the vertical displacement at the midpoint on lower surface of the beam \((x = 40, y = 0)\), using Monte Carlo (MC) and generalized polynomial chaos (GPC) of order \( p = 2 \) and 3. For MC simulations, we use 10,000 realizations of \( E \) (generated according to the uniform distribution), and corresponding to each realization the deterministic problem is solved. The PDFs are generated in a similar manner as described in Section 4.2.3 for electrostatics, using 10 equally spaced bins. As can be seen, there is a good agreement between the PDFs obtained using MC and GPC. We also notice that the PDF obtained using GPC, converges to the MC result, as we increase the order of the expansion from \( p = 2 \) to \( p = 3 \). The mean vertical displacement of the beam, with the error bars (based on standard deviation) is plotted in Fig. 5b. The mean and standard deviation for the vertical displacement at the midpoint, using MC and GPC of various orders are tabulated in Table 2.

4.4. Algorithm for stochastic coupled electromechanical problem

Using the stochastic formulation for electrostatics and large deformation elasticity, the algorithm for stochastic coupled electromechanical problem, applicable to static analysis of MEMS is given in Algorithm 1.

**Algorithm 1.** Stochastic coupled electromechanical problem

1. Identify uncertain parameters (material properties and geometrical parameters) and represent them in terms of independent random variables \( \xi = [\xi_1, \xi_2, \ldots, \xi_n]^T \), such that \( n \) represents the dimension of the random domain.
2. Represent undeformed (or mean) initial domains as \( \Omega_i \) with boundary \( d\Omega_i \), and discretize.
3. Model geometrical uncertainties in initial configuration as random deformation field \( u(X, \theta) \) applied to conductors defined using mean configuration.
4. Initialization Set \( i = 0 \) and the spectral modes for the uncertain surface charge density \( [\xi_k] = 0 \).
5. Expand uncertain material tensor \( \mathcal{E}(\theta) = \sum_{k=0}^{N} \mathcal{E}_k \Psi_k \) and uncertain deformation field \( [u(X, \theta)] = [u_k] \Psi_k \) in terms of GPC basis functions, where \( N + 1 \) is the total number of terms considered.

| Mean and standard deviation for midpoint vertical displacement \((v_E = 0.2)\) |
|------------------|------------------|------------------|
|                  | MC               | GPC2             | GPC3             |
| Mean (µm)        | -1.47666         | -1.47665         | -1.47666         |
| Std (µm)         | 0.07806          | 0.07804          | 0.07806          |

![Fig. 5. Statistics of the vertical displacement for doubly clamped beam using MC and GPC for a 20% variation in Young’s modulus \( E \).](attachment:image.png)
Stochastic electrostatic analysis
7: repeat
a. Using \( \mathbf{u} = [u_i]^{k+1} \), solve for the spectral modes 
\( [\sigma_k]^{i+1} \), such that the uncertain surface charge density is given as 
\( \sigma(x, \theta) = \sum_{i=0}^{N} \sigma_k^{i+1} \psi_k \).

b. Using \( \sigma^{i+1} = [\sigma_k]^{i+1} \psi_k \), expand the electrostatic pressure 
\( \mathbf{H}^{i+1} = \sum_{k=0}^{N} [H_k]^{i+1} \psi_k \).

8: Stochastic mechanical analysis
Using \( \mathbf{C} = \mathbf{C} \psi_k, \mathbf{u} = [u_i]^{k+1} \psi_k \) and 
\( \mathbf{H}^{i+1} = [H_k]^{i+1} \psi_k \), compute the spectral modes 
\( [u_i]^{i+1} \), such that the uncertain deformation field is represented as 
\( \mathbf{u}^{i+1} = \sum_{k=0}^{N} [u_k]^{i+1} \psi_k \).

9: Update \( i = i + 1 \).
10: until A self-consistent solution is obtained, such that 
\( ||[u_i]^{i+1} - [u_i]^i|| < \text{tol} \) and 
\( ||[\sigma_k]^{i+1} - [\sigma_k]^i|| < \text{tol}, \forall k \).
11: Post-processing Compute the statistics (such as mean and standard deviation) of the deformation 
\( \mathbf{u}(X, \theta) \) and surface charge density \( \sigma(X, \theta) \).

5. Numerical results
In this section, we present a few numerical examples to demonstrate the methodology developed in the previous section, to quantify the effect of uncertain material properties and geometrical parameters on the performance of electrostatically actuated MEMS devices.

In the first example we consider a micro-switch, modeled by a cantilever beam over a ground plane, and study the effect of uncertain Young’s modulus and gap between the beam and the ground plane, on its actuation and pull-in behavior. Secondly, we consider a comb drive structure to highlight the propagation of uncertainties from the mechanical to the electrostatic energy domain. For both the examples, we assume that the uncertain material and geometrical parameters are defined using uniformly distributed random variables. This choice of uniform distribution is inspired, in part, by the lack of knowledge of the actual distribution suitable to model these uncertain parameters for MEMS and also by the fact that the variation in these parameters is often reported in terms of error bars. Hence, the most straightforward way is to assume that the uncertain parameter is uniformly distributed over the given range.

5.1. MEMS switch
We consider a MEMS switch, modeled by a cantilever beam, which is 80 \( \mu \)m long, 1 \( \mu \)m thick and 10 \( \mu \)m wide, located over a ground plane. As the applied potential difference \( V \) between the beam and the ground plane is increased, the beam increasingly deflects towards the ground plane. At a voltage (known as the pull-in voltage) when the electrostatic force can no longer be balanced by the restoring elastic force, the beam finally collapses onto the ground plane. In order to design reliable switches, it is required that we consider the variations in material properties and geometrical parameters, and quantify their effect on device performance parameters such as deflection and pull-in voltage, etc. Specifically, we consider the effect of uncertain Young’s modulus \( E \) and the initial gap \( g \) between the beam and the ground plane, which are given as:

\[
E = E_0(1 + \nu_E \xi_1), \quad g = g_0(1 + \nu_g \xi_2),
\]

where \( E_0 = 169 \) GPa and \( g_0 = 1 \) \( \mu \)m are the mean values, \( \nu_E = \nu_g = 0.1 \), which represents a 10% variation in both parameters, and \( \xi_1, \xi_2 \) are independent uniformly distributed random variables in \([-1, 1]\). Thus, we employ a two-dimensional generalized polynomial chaos expansion, based on Legendre polynomials, for all unknown random quantities. For example, the displacement field \( \mathbf{u}(X, \theta) = [u(X, \theta)] \) is expanded as:

![Probability density function of tip displacement using GPC and MC.](image1)

![Vertical displacement with error bars, \( p = 3 \).](image2)

Fig. 6. Statistics of the vertical displacement for cantilever beam, \( [\nu_E = \nu_g = 0.1, V = 7.0 \text{V}] \).
\[ \mathbf{u}(\mathbf{X}, \theta) = \mathbf{u}_0 + \mathbf{u}_1 \zeta_1 + \mathbf{u}_2 \zeta_2 + \mathbf{u}_3 \frac{1}{2}(3\zeta_1^2 - 1) + \mathbf{u}_4 \zeta_1 \zeta_2 + \mathbf{u}_5 \frac{1}{2}(3\zeta_2^2 - 1) + \cdots, \] (50)

Using the algorithm given in Section 4.4, we solve for the spectral modes \{\mathbf{u}_i\}.

### 5.1.1. Verification using MC simulations

Having computed the spectral modes \{\mathbf{u}_i\}, the vertical tip displacement (which is a random variable), is given as

\[ v_{\text{tip}}(\theta) = v(\mathbf{X}_{\text{tip}}, \theta). \] (51)

The results obtained using GPC are verified against the statistics obtained using Monte Carlo (MC) simulations. For MC, we use 10,000 samples each for \( \zeta_1 \) and \( \zeta_2 \) generated according to the uniform distribution, such that \( \zeta_1 \) and \( \zeta_2 \) are independent. For each \( (\zeta_1, \zeta_2) \) pair, we solve the deterministic coupled electromechanical problem, and compute the values \{\( v_{\text{tip}} \)\}. In Fig. 6a, we plot the PDF for the vertical tip displacement, using MC and GPC expansion of order \( p = 2 \) and \( 3 \), for an applied potential difference \( V = 7.0 \) V. The close agreement between the PDFs obtained using MC and GPC verifies the statistics obtained using generalized polynomial chaos. In Fig. 6b, we plot the mean vertical deflection with error bars, based on the standard deviation obtained using GPC with \( p = 3 \). The mean and standard deviation values for the tip deflection using MC and GPC of various orders are tabulated in Table 3.

### 5.1.2. Pull-in behavior

The pull-in voltage \( V_p \) and the tip deflection \( v_{\text{tip}} \) at \( V = V_p \) (deflection just before pull-in), computed by solving the deterministic coupled problem for various values of \( E \) and \( g \) are tabulated in Table 4. In Fig. 7, we plot the mean tip deflection and error bars based on standard deviation, computed using GPC expansion of order \( p = 2 \), with different applied potential difference. The corresponding mean and standard deviation values are tabulated in Table 5.

<table>
<thead>
<tr>
<th>( V ) (V)</th>
<th>( \bar{v} ) (( \mu m ))</th>
<th>( \sigma ) (( \mu m ))</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>0.00220</td>
<td>0.0003</td>
</tr>
<tr>
<td>2.0</td>
<td>0.0091</td>
<td>0.0013</td>
</tr>
<tr>
<td>3.0</td>
<td>0.0207</td>
<td>0.0031</td>
</tr>
<tr>
<td>4.0</td>
<td>0.0379</td>
<td>0.0059</td>
</tr>
<tr>
<td>5.0</td>
<td>0.0615</td>
<td>0.0101</td>
</tr>
<tr>
<td>6.0</td>
<td>0.0935</td>
<td>0.0169</td>
</tr>
<tr>
<td>7.0</td>
<td>0.1381</td>
<td>0.0292</td>
</tr>
<tr>
<td>7.8</td>
<td>0.1932</td>
<td>0.0548</td>
</tr>
</tbody>
</table>

In addition to these statistics, we also plot the pull-in curves obtained by solving the deterministic problem for three separate samples given by \( S_0 : (E = E_0, g = g_0), S_1 : (E = 0.9E_0, g = 0.9g_0) \) and \( S_2 : (E = 1.1E_0, g = 1.1g_0) \). As can be easily seen that the realizations \( S_1 \) and \( S_2 \) mark the two boundaries for the pull-in curve and are indicative of the worst-case behavior of the switch. Thus, for an applied voltage \( V < 7.3 \) V, for all possible values \((E, g)\), the vertical tip deflection of the beam would be between the values indicated by these two curves. One could use the information regarding the worst-case behavior to design reliable switches, but such designs may be over-conservative. However, the information obtained in terms of the error bars by the stochastic analysis, may be used to design reliable and efficient devices.

We also observe that the mean deflection is greater than the deflection obtained by solving the deterministic problem, assuming mean values for the uncertain parameters \( (S_0) \). This indicates that the actual mean performance of the device could be significantly different than the performance expected using mean values for all uncertain parameters.

As the applied voltage increases from 7.3 V to 11.4 V, the probability of pull-in increases from 0 to 1. We note that the pull-in phenomenon is manifested as a discontinuity in the random space, since for the realizations which
leads to pull-in, the value of the tip deflection $v_{\text{tip}}$ is equal to the gap $g$. It is for this very reason that the approach using generalized polynomial chaos expansion, based on global Legendre polynomials fails to converge \cite{27,28}, as the probability of pull-in becomes significant. For expansion order $p=2$, we obtain convergent results for $V \leq 7.8\, V$. In order to obtain statistics beyond this voltage, we need to use local basis functions in the stochastic domain \cite{27,29–31}, which would be considered in future. This work however, presents a methodology to handle uncertainties in a multiphysics problem. We also demonstrate that the generalized polynomial chaos can be used effectively to obtain statistics verifiable using MC simulations, as long as the assumption about the solution being smooth in the random parameter space holds true.

The PDFs obtained using GPC expansion of order $p=2$, for various applied voltages are shown in Fig. 8. As can be seen, the PDFs get increasingly skewed as the applied voltage increases.

5.1.3. Sensitivity analysis

The objective of the stochastic analysis is twofold – firstly, to quantify the effect of various uncertain design parameters on relevant performance parameters. Secondly, to use that information to identify critical design parameters, which one may need to control carefully, in order to design reliable devices. This can be achieved by considering the sensitivity of the relevant performance parameter to the uncertain design parameters. The GPC expansion as given in Eq. (50), can be directly used to compute the sensitivity of the tip deflection to the variations in Young’s modulus $s_E = \frac{1}{E \xi_{E}}$ and gap $s_g = \frac{1}{g \xi_{g}}$. We note that these sensitivities are functions of $\xi$ and their mean has been plotted in Fig. 9. From this figure, we can conclude that the tip deflection is more sensitive to the same variation (fraction around the mean) in gap as opposed to variation in the Young’s modulus. Also this effect is more pronounced at applied voltages close to the pull-in voltage.

5.2. Comb drive

Comb drives form an important class of MEMS devices and have been used in widespread applications such as micro-accelerometers, hard disk actuators and position controllers. We consider a comb drive structure as shown in Fig. 10, which consists of a movable structure supported on folded springs and a set of interdigitated teeth. An electrostatic force is generated when a potential difference is applied between the fixed teeth and the movable teeth attached to the movable structure, which provides a vertical movement. The effect of various geometrical features on the design of a comb drive has been studied in \cite{6} using Monte Carlo method incorporated in the ANSYS probabilistic design system (ANSYS/PDS).

We consider this example to demonstrate the propagation of uncertainties from the mechanical to the electrostatic energy domain. Specifically, we quantify the effect of uncertainty in the Young’s modulus of the comb structure on the capacitance between the fixed and movable teeth, in the deformed configuration. We consider the Young’s modulus $E$ to be a uniformly distributed random
variable, as before, given by \( E = E_0 (1 + \nu_E \xi) \), where \( E_0 = 200 \) GPa (for nickel), \( \nu_E = 0.1 \) and \( \xi \) is a uniformly distributed random variable in \([-1, 1]\). The Poisson’s ratio is set to be 0.3.

In Fig. 11a, we plot the PDFs for capacitance between the fixed and the movable tooth using MC simulations (based on 5000 samples) and GPC expansion of order \( p = 3 \) and 4, for an applied potential difference \( V = 4.6 \) V. We also plot the PDFs for vertical displacement at the upper right corner of the movable tooth in Fig. 11b. The corresponding mean and standard deviation values for capacitance and deflection are given in Tables 6 and 7, respectively.

### Table 6
Mean and standard deviation for capacitance, \( V = 4.6 \) V, \( (\nu_E = 0.1) \)

<table>
<thead>
<tr>
<th></th>
<th>MC</th>
<th>GPC3</th>
<th>GPC4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>( 7.2000 )</td>
<td>( 7.2010 )</td>
<td>( 7.2014 )</td>
</tr>
<tr>
<td>Std</td>
<td>( 0.1918 )</td>
<td>( 0.1885 )</td>
<td>( 0.1894 )</td>
</tr>
</tbody>
</table>

### Table 7
Mean and standard deviation for vertical deflection, \( V = 4.6 \) V, \( (\nu_E = 0.1) \)

<table>
<thead>
<tr>
<th></th>
<th>MC</th>
<th>GPC3</th>
<th>GPC4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>( 0.1541 )</td>
<td>( 0.1542 )</td>
<td>( 0.1542 )</td>
</tr>
<tr>
<td>Std</td>
<td>( 0.0208 )</td>
<td>( 0.0206 )</td>
<td>( 0.0207 )</td>
</tr>
</tbody>
</table>

### 6. Conclusions

This work presented a stochastic framework to quantify the effect of uncertain material properties and geometrical parameters, during the static analysis of electrostatic MEMS. The proposed approach comprises of stochastic electrostatic and mechanical analyses. The stochastic analysis is based on a stochastic Lagrangian approach, where, in addition to uncertain input parame-
ters and unknown field variables, the random deformed configuration is expanded in terms of GPC basis functions. The spectral modes for the unknown field variables are obtained using Galerkin projection in the space spanned by GPC basis functions. The stochastic mechanical and electrostatic analyses are performed in a self-consistent manner to obtain the random deformation of the MEMS structures. To the best of our knowledge, it is for the first time that a framework has been presented for accurate stochastic modeling of coupled electromechanical interaction.

We considered two numerical examples—first, a MEMS switch, as modeled by a cantilever beam located over a ground plane, and studied the effect of uncertain Young’s modulus and gap between the electrodes, on the actuation properties and pull-in behavior. Second, we considered a comb drive structure and studied the propagation of uncertainty from elastic energy domain to the electrostatic, by quantifying the effect of uncertain Young’s modulus on the capacitance between the fixed and movable teeth. Using Monte Carlo simulations, we verified that as long as the solution is smooth in the random parameter space, the proposed methodology accurately predicts the probability density function and statistics such as mean and standard deviation, of various relevant output parameters. We also demonstrated that the information obtained during the uncertainty quantification process, can then be used effectively to identify critical design parameters, based on a sensitivity analysis. In the past, it has been shown that the polynomial chaos approach can be up to an order of magnitude faster than MC simulations for reasonable dimensionality [32,27]. Thus, the proposed methodology can be used effectively to quantify the effect of variations in geometrical and material properties on the performance of electrostatic MEMS devices. Further, this information can be used to identify critical design parameters which can then be controlled, leading to reliable and efficient devices.

Acknowledgement

This work is supported by the National Science Foundation under Grant No. 0601479, and by DARPA/MTO.

Appendix A. Computations for stochastic large deformation analysis

For the deterministic case, the global stiffness matrix $\mathbf{K}$ given in Eq. (35) can be written in terms of the elemental contributions as $\mathbf{K} = \sum_{e=1}^{N_e} \mathbf{k}_e$, such that $\mathbf{k}_e$ is the contribution from element $e$ and $N_e$ represents the total number of elements [33]. The elemental matrix $\mathbf{k}$ (dropping the superscript $e$ for brevity) can be decomposed into the material part $\mathbf{k}^m$ and the geometric part $\mathbf{k}^g$ as follows:

$$\mathbf{k} = \mathbf{k}^m + \mathbf{k}^g, \quad \mathbf{k}^m = [\mathbf{k}^m_{\phi_p}], \quad \mathbf{k}^g = [\mathbf{k}^g_{\phi_p}],$$  \hspace{1cm} (A.1)

$$\mathbf{k}^m_{\phi_p} = \int_{\Omega^e} \mathbf{B}_{\phi_p}^{T} \mathbf{C}_{\phi_p} \mathbf{B}_{\phi_p} \, d\Omega,$$
$$\mathbf{k}^g_{\phi_p} = \int_{\Omega^e} \mathbf{B}_{\phi_p}^{T} \mathbf{S}_{\phi_p} \mathbf{B}_{\phi_p} \, d\Omega,$$  \hspace{1cm} (A.2)

where, the geometric part is defined using,

$$\mathbf{S} = \left[ \begin{array}{c} \mathbf{S} \\ 0 \end{array} \right], \quad \mathbf{B}_{\phi_p} = \left[ \begin{array}{ccc} [\nabla N_{e}^1]_{2 \times 1} & 0 \\ 0 & [\nabla N_{e}^1]_{2 \times 1} \end{array} \right]_{4 \times 2},$$  \hspace{1cm} (A.3)

and the material part is defined using,

$$\mathbf{B}_{\phi_p} = \left[ \begin{array}{ccc} \phi_{1,1} N_{x,1} & \phi_{2,1} N_{x,1} \\ \phi_{1,2} N_{x,2} & \phi_{2,2} N_{x,2} \\ \phi_{1,1} N_{x,2} + \phi_{1,2} N_{x,1} & \phi_{1,1} N_{x,2} + \phi_{2,2} N_{x,1} \end{array} \right]_{3 \times 2},$$  \hspace{1cm} (A.4)

The residual vector $\mathbf{r}(\mathbf{d})$ is defined as $\mathbf{r}(\mathbf{d}) = \sum_{e=1}^{N_e} \mathbf{r}_e$, where the elemental contribution $\mathbf{r}_e = [\mathbf{r}_e]$ is given as,

$$\mathbf{r}_e = \int_{\Omega^e} \mathbf{B}_{\phi_p}^{T} \tilde{\mathbf{S}} \, d\Omega - \int_{\Gamma^e} \mathbf{N}_{x} \mathbf{H} \mathbf{d} \mathbf{G}, \quad \tilde{\mathbf{S}} = \left[ \begin{array}{c} S_{11} \\ S_{22} \\ S_{12} \end{array} \right]_{3 \times 1},$$  \hspace{1cm} (A.5)

Now, for the stochastic case, we seek to expand the random stiffness matrix $\mathbf{K}$ and random residual vector $\mathbf{R}$, in terms of the GPC basis functions, as given by Eq. (43). We employ the following discretization for $\mathbf{u}(\mathbf{X}, \mathbf{\theta})$,

$$\mathbf{u}(\mathbf{X}, \mathbf{\theta}) = \sum_{x=1}^{N_x} \mathbf{N}_{x}^{\mathbf{X}}(\mathbf{X}) \mathbf{d}_{x}(\mathbf{\theta})$$
$$= \sum_{x=1}^{N_x} \mathbf{N}_{x}^{\mathbf{X}}(\mathbf{X}) \left( \sum_{j=0}^{N} \mathbf{d}_{j}^{\mathbf{X}} \psi_{j}(\mathbf{\theta}) \right) = \sum_{j=0}^{N} \mathbf{u}^{j} \psi_{j}(\mathbf{\theta}),$$  \hspace{1cm} (A.6)

where $\mathbf{u}^{j} = \sum_{x=1}^{N_x} \mathbf{N}_{x}^{\mathbf{X}}(\mathbf{X}) \mathbf{d}_{j}^{\mathbf{X}}$. The deformation gradient $\mathbf{F}(\mathbf{X}, \mathbf{\theta})$ is given as

$$\mathbf{F}(\mathbf{X}, \mathbf{\theta}) = \left[ \begin{array}{cc} 1 + u_{1,1}^{j} \psi_{j} & u_{1,2}^{j} \psi_{j} \\ u_{2,1}^{j} \psi_{j} & 1 + u_{2,2}^{j} \psi_{j} \end{array} \right] = \mathbf{F}_{j} \psi_{j},$$  \hspace{1cm} (A.7)

$$\mathbf{F}_{0} = \left[ \begin{array}{cc} 1 + u_{1,1}^{0} & u_{1,2}^{0} \\ u_{2,1}^{0} & 1 + u_{2,2}^{0} \end{array} \right], \quad \mathbf{F}_{j} = \left[ \begin{array}{cc} u_{1,1}^{j} & u_{1,2}^{j} \\ u_{2,1}^{j} & u_{2,2}^{j} \end{array} \right], \quad \forall j > 0.$$  \hspace{1cm} (A.8)

Now, $\mathbf{S} = \mathbf{C} \mathbf{E}$, where $\mathbf{C} = \mathbf{C}_{\mathbf{K}} \mathbf{K}$ for uncertain material properties and $\mathbf{E} = \frac{1}{2} (\mathbf{F}^{T} \mathbf{F} - \mathbf{I})$, can be expanded as follows:
\( A = F^T F = A_k \Psi_k \), where \( A_k = \sum_i \sum_j F_i^j F_j^i e_{ijk} / d_{ijk} \), \( (A.10) \)

\( E = E_\theta \Psi_k \), where \( E_\theta = \frac{1}{2} (A_0 - I) \), \( E_\theta = \frac{1}{2} A_k \), \( \forall k > 0 \), \( \quad (A.11) \)

\( S = S_k \Psi_k \), where \( S_k = \sum_i \sum_j \mathcal{E}_i e_{ijk} / d_{ijk} \). \( (A.12) \)

Using the expansion for \( S \), the geometric part of \( k \) (elemental) can be expanded as follows:

\( k^\xi = k^\xi J \Psi_i \), where \( k^\xi = \left[ k^\xi \right]_{\phi} \). \( (A.13) \)

\[
[k^\xi]_{\phi} = \int_{\Omega} [\mathbf{B}_{NL}]^T \mathbf{S}_i [\mathbf{B}_{NL}]_\phi d\Omega,
\]

\[
\mathbf{S}_i = \begin{bmatrix} [S] & 0 \\ 0 & [S] \end{bmatrix}.
\] \( (A.14) \)

In order to expand the material part \( k^m \) we first need to expand \( \mathbf{B}_{l_i} \), as given by \( (A.5) \):

\[
[\mathbf{B}_{l_i}]_x = \left[ \mathbf{B}_{l_i}^0 \right] \Psi_i,
\] \( \quad (A.15) \)

\[
[\mathbf{B}_{l_i}]_x^0 = \begin{bmatrix} N_{x,1} & 0 \\ 0 & N_{x,2} \\ N_{x,2} & N_{x,1} \end{bmatrix} \left[ \begin{array}{c} u_{1,1}^0 N_{x,1} \\ u_{1,2}^0 N_{x,2} \\ u_{1,1}^0 N_{x,2} + u_{1,2}^0 N_{x,1} \end{array} \right],
\]

\[
+ \left[ \begin{array}{c} u_{2,1}^0 N_{x,1} \\ u_{2,2}^0 N_{x,2} \\ u_{2,1}^0 N_{x,2} + u_{2,2}^0 N_{x,1} \end{array} \right],
\] \( \quad (A.16) \)

\[
[\mathbf{B}_{l_i}]_x^m = \begin{bmatrix} u_{i,1}^m N_{x,1} \\ u_{i,2}^m N_{x,2} \\ u_{i,1}^m N_{x,2} + u_{i,2}^m N_{x,1} \end{bmatrix},
\] \( \quad (A.17) \)

The material part \( k^m \) can then be expanded as follows: We first define \( \mathbf{D}_\phi = [\mathbf{B}_{l_i}]_x^m [\mathbf{B}_{l_i}]_x^0 \), and expand \( \mathbf{D}_\phi = \mathbf{D}_\phi^\xi \mathbf{J}_\phi \mathbf{D}_\phi^m \) using the pseudo-spectral technique described earlier. Thus,

\[
k^m = k^m J \Psi_i,
\] \( \quad (A.18) \)

\[
[k^m]_{\phi} = \int_{\Omega} [\mathbf{D}_\phi^m] d\Omega.
\] \( (A.19) \)

From \( (A.13) \) and \( (A.18) \), we can write

\[
\mathbf{K} = \mathbf{K} / J \Psi_i,
\] \( \quad (A.20) \)

\[
\mathbf{K}_i = \sum_{e=1}^{N_e} \mathbf{k}_e^i,
\] \( \quad (A.21) \)

The residual vector \( \mathbf{R}(\mathbf{d}) \) can be expanded as follows:

\[
\mathbf{R}(\mathbf{d}) = \mathbf{R} / (\Psi_i),
\] \( \quad (A.21) \)

where the elemental contribution \( \mathbf{r}_i = [\mathbf{r}_i]_x \) (dropping the superscript \( e \)) can then be computed using

\[
[r_i]_x = \int_{\Gamma} [\mathbf{B}_l]_x^m \mathbf{S}_i d\Omega - \int_{\Gamma} N_a \mathbf{H} d\Gamma,
\] \( (A.22) \)

\[
[[\mathbf{B}_l]_x^m \mathbf{S}_i] = \sum_i \sum_j [[\mathbf{B}_l]_x^m]_j \mathbf{S}_j \left( \frac{e_{ijk}}{d_{ijk}} \right),
\] \( (A.23) \)

where \( \mathbf{S} = \mathbf{S}_j \mathbf{J}_j \mathbf{S}_i \) can be expanded using the expansion for \( \mathbf{S} \).

The applied traction \( \mathbf{H} \) can be expanded depending on its form. Specifically, when the applied traction comes from electrostatic pressure, it is given as:

\[
\mathbf{H} = J P \mathbf{F}^T \mathbf{N},
\] \( \quad (A.24) \)

where \( J = \text{det}[\mathbf{F}] \), \( \mathbf{N} \) is the unit outward normal in the undeformed configuration and \( P_e = \frac{\sigma}{\epsilon} \) is the electrostatic pressure, such that \( \sigma \) is the surface charge density. In such situation, \( \mathbf{H} \) can be expanded as follows:

\[
\mathbf{F}^T = \frac{1}{J} \begin{bmatrix} F_{2,2} & -F_{2,1} \\ -F_{1,2} & F_{1,1} \end{bmatrix} = \frac{1}{J} \mathbf{F}_x \mathbf{J} \mathbf{Psi}_i,
\]

\[
\mathbf{F}_x = \begin{bmatrix} (F_{x})_{2,2} & -(F_{x})_{2,1} \\ -(F_{x})_{1,2} & (F_{x})_{1,1} \end{bmatrix},
\] \( \quad (A.25) \)

\[
\sigma = \sigma_x \Psi_i \Rightarrow P_e = \frac{1}{2\epsilon} \left( \sum_i \sum_j \sigma_x \sigma_j \left( \frac{e_{ijk}}{d_{ijk}} \right) \right) \mathbf{Psi}_i = P_{e_k} \mathbf{Psi}_i,
\] \( \quad (A.26) \)

\[
\mathbf{H}_k = \sum_i \sum_j P_{e_k} \mathbf{N}_i \left( \frac{e_{ijk}}{d_{ijk}} \right).
\] \( \quad (A.27) \)

References


